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Gravity Level Influence on a Laterally Heated Ferrofluid Submitted to an Oblique Strong Magnetic Field

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Magnetic Liquid / Laminar Flows

A horizontal ferrofluid layer is submitted to a lateral heating and to a strong oblique magnetic field. For a thin enough layer, the steady state solution is the product of the classical solution corresponding to the usual Newtonian fluid submitted to a lateral gradient of temperature, times a modulating factor. This last accounts for the inclination and for the ratio Ra_m/Ra, where Ra is the Rayleigh number due to buoyancy and Ra_m the magnetic Rayleigh number due to the Kelvin magnetic forces. Physically, fundamental modifications of the temperature and velocity profile intervene when $Ra_m/Ra \gg 1$, since, as function of the inclination of the magnetic field, the direction of motion in the infinitely elongated cells can become inverted.

1. Introduction

Huang *et al.* [1] considered an oblique magnetic field and became interested in the stability problem of a *vertical* temperature gradient, to explain an experiment [2], extending thus the Rayleigh–Bénard study started by Finlayson [3]. In all cases, on earth and also in microgravity [4], the reference state was *heat conductive and motionless*. Birikh and others [5–7] initiated the thermo-convective problem, induced by a *lateral* gradient of temperature heating a Newtonian liquid layer so that a profile of velocity and of tempera-

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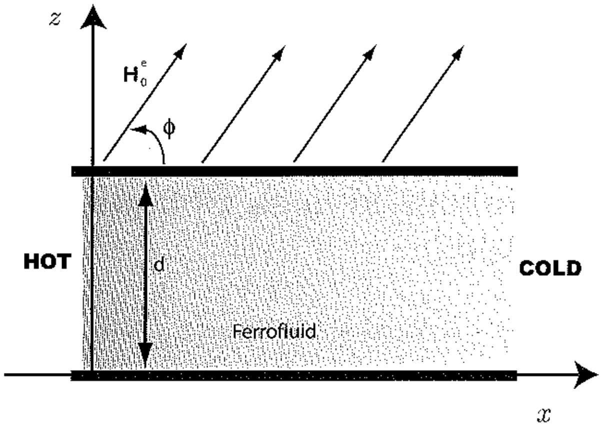

Fig. 1. Laterally heated ferrofluid layer submitted to an inclined magnetic field $\mathbf{H}_0^e$.

ture exists in the reference case. A stability study should start from that *steady motion.*

Here, now, a ferrofluid of width d, bounded by two rigid plates, is submitted to an inclined strong magnetic field and to a lateral gradient of temperature. We limit ourselves to a 2D description, and consider the central region of a very elongated and thin layer, ignoring the lateral walls influence, to describe the reference steady state [8], as this leads to surprising results in microgravity.

2. Formulation of the problem

This layer of ferrofluid is submitted to a time independent exterior constant magnetic field, given by $\mathbf{H}_0^e = H_0^e(\cos\phi \mathbf{1}_x + \sin\phi \mathbf{1}_z) = H_0^e \mathbf{1}_H$ where $\mathbf{1}_z$ is the upward unit normal, $\mathbf{1}_x$ is the horizontal unit taken along the lower border and $\mathbf{1}_H = \mathbf{H}/\sqrt{\mathbf{H}\cdot\mathbf{H}}$ (see Fig. 1). We describe the ferrofluid, as a super paramagnetic gas [9], where the magnetization $\mathbf{M}$ is collinear with the magnetic field. We will consider here very strong magnetic fields, so that the magnetization does not more depend on the magnetic field and the magnetization state equation $M = M(T)$ will be [3, 9, 10]:

$$\mathbf{M} = M\mathbf{1}_H \quad \text{where} \quad M = M(T_0, H_0) - K(T - T_0)\,, \tag{1}$$

near to the reference state $\mathbf{M}(T_0, H_0)$. The pyromagnetic coefficient $K = \alpha M(T_0)$ is assumed constant, with α being the thermal expansion coefficient [9, 10] and we apply the inductive assumption to the Maxwell equations [3, 9, 10] which become:

$$\nabla \times \mathbf{H} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{H} = \alpha M(T_0)\mathbf{1}_H \cdot \nabla T\,. \tag{2}$$

The reference steady motion is a consequence of our basic assumption: *there exists a basic horizontal flow depending on height.* From this and from the ferrofluid incompressibility, follows that there is no net mass flux across a vertical section

$$\int_0^d u(z)\,\mathrm{d}z = 0 \quad \text{since} \quad \mathbf{u} = u(z)\mathbf{1}_x\,. \tag{3}$$

Both the lower and upper borders are rigid non-magnetic plates and the ferrofluid is a viscous liquid, adhering to a solid boundary, so that $u(z) = 0$ at $z = 0, d$.

The Boussinesq approximation of the momentum balance equation is then [3, 9, 10]:

$$0 = -\nabla p + \mu_0 M \nabla H + \eta \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\mathbf{1}_x - \rho_0 g\left[1 - \alpha\left(T - T_0\right)\right]\mathbf{1}_z\,, \tag{4}$$

where ρ_0 is the density at the reference temperature T_0 of the liquid layer and η the kinematic viscosity. We neglect thermomagnetophoresis [11]. From Eq. (4), we obtain:

$$\eta \frac{\mathrm{d}^3 u}{\mathrm{d}z^3} = \rho_0 g \alpha \frac{\partial T}{\partial x} - K\mu_0\left[\frac{\partial T}{\partial x}\frac{\partial H}{\partial z} - \frac{\partial T}{\partial z}\frac{\partial H}{\partial x}\right] = G\,, \tag{5}$$

where $G = G(z)$ to satisfy the basic assumption (3). An important consequence is then that $H = H(z)$ [8].

The energy balance equation for the ferrofluid, derived by Finlayson [3], is

$$\rho c_p u \frac{\partial T}{\partial x} - \mu_0 T K u \frac{\partial H}{\partial x} = \lambda \nabla^2 T\,, \tag{6}$$

where c_p is the heat capacity of the ferrofluid and λ is the thermal conductivity. To complete the problem, we need the following boundary conditions: from the Maxwell equations (2), the normal component of $\mathbf{H}+\mathbf{M}$ and the tangential component of the magnetic field $\mathbf{H}$ have to be continuous across the top and the bottom boundaries [1, 3, 9, 12]. This means we have $B_z = H_0^e \sin\phi$, and $H_x = H_0^e \cos\phi$, all along both horizontal boundaries at $z = 0, d$. Also, along those same boundaries we assume that $T = T_{\mathrm{ref}} - \beta x$ where T_{ref} is the temperature at $x = 0$, (β here is positive [5]).

3. The dimensionless steady problem

We will discuss only the steady solution of the problem rewritten in a dimensionless form. We use d to scale both spatial coordinates so that $0 \leq z \leq 1$.

Thus, βd scales the temperature and κ/d, the velocity. The problem will be to find $t(z) = -x + (T_{\text{ref}} - T)/(\beta d)$ and the dimensionless magnetic field $\mathbf{h} = h(z)\mathbf{1_H} = \mathbf{H}/H_0^e$ or $\mathbf{h} = \cos\phi \mathbf{1}_x + h_z(z)\mathbf{1}_z$. One has $t(0) = 0$ as well as $t(1) = 0$. We obtain from the energy balance equation:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}z^2} - u(z) = 0\,, \tag{7}$$

and from the dimensionless equivalent of the momentum balance (5):

$$\frac{\mathrm{d}^3 u}{\mathrm{d}z^3} = -Ra\left\{1 + \frac{Ra_{\text{m}}}{Ra}\frac{\cos\phi\sqrt{h^2 - \cos^2\phi} + (\mathrm{d}t/\mathrm{d}z)(h^2 - \cos^2\phi)}{h^2}\right\}. \tag{8}$$

The adimensionalisation process introduces two positive dimensionless numbers, Ra and Ra_{m}, where $Ra = (\rho_0 \alpha g \beta d^4)/(\eta\kappa)$ is the classical Rayleigh number [1], and where Ra_{m} is the usual magnetic Rayleigh number [9, 10] $Ra_{\text{m}} = \beta^2 \mu_0 K^2 d^4/\eta\kappa$. In last equation, the dimensionless magnetic field is [8]

$$h = 1 - \epsilon_H h_1(z)\,. \tag{9}$$

The quantity $\epsilon_H = K\beta d/H_0^e$ is small, reflecting the strong magnetic field assumption, $K\beta d$ scaling the total variation of the magnetization $M(T)$ across the layer in terms of the external field H_0^e. Ra_{m}/Ra is independent both on the depth d of the layer and on the value of the applied magnetic field H_0^e. Eqs. (7) and (8) have to be completed by the dimensionless expressions of the boundary conditions along the rigid plates $t = u = 0$ at $z = 0, 1$.

4. The solution as a power series in ϵ_H

Due to the non-linearity of Eq. (8), one cannot obtain the analytical profiles of $u(z), t(z), h(z)$, for any given oblique inclination. Still, some general conclusions can be obtained from a carefull study of the dimensionless parameters. We consider the well known physical data of EMG 901 [12] (see Table 1). The boiling point of EMG 901 is around 250 °C and we know EMG 901 to be a liquid, at normal temperature. The layer total length L must be large enough $d \ll L$, to neglect the influence of the lateral walls, in the middle region [6, 7]. Then, assuming $L = 15$ cm and $d < 10$ mm, $\beta \leq 1$ K/mm, so that the total drop of temperature $(T_{\text{hot}} - T_{\text{cold}})/L \approx \beta$, from one vertical end wall to the other, should be less than 150 °C. For EMG 901, the magnetization at saturation is $M_{\text{sat}} = 4.8 \times 10^4$ A/m [12], which should be smaller than the exterior magnetic field H_0^e to obey the strong field assumption. Then, ϵ_H is less than $0.001 \ll 1$. We will thus expand $u(z), h(z)$ and $t(z)$ in a power series of ϵ_H.

Table 1. Fluid properties for EMG 901 [12] and the dimensionless numbers.

symbol	value	units	quantity
ρ	1530	kg/m^3	density
ν	6.5×10^{-6}	m^2/s	kinematic viscosity
κ	8.2×10^{-8}	m^2/s	heat diffusivity
α	6.0×10^{-4}	K^{-1}	thermal expansion
M_{sat}	4.8×10^4	A/m	saturation magnetization
K	29	A/K m	pyromagnetic coefficient
d	1–10	mm	variable width
β	0.1–1.0	K/mm	variable lateral gradient
g	9.8×10^{-n}	$m\,s^{-2}$	variable gravity $6 \geq n \geq 0$
Ra	$11 \times 10^{-n} \times d^4\beta$		thermal Rayleigh number
Ra_m	$1.28 \times d^4\beta^2$		magnetic Rayleigh number [10]
ϵ_H	$29 \times d\beta/H_0^e$		coupling of magn. and temp.

5. The zero order term

This leads to several conclusions on the zero order solution $h_0(z)$, $u_0(z)$, $t_0(z)$. First, the magnetic field leading order term is $h_0(z) = 1$ and secondly, the leading order term of dt/dz appearing in the RHS of Eq. (8), is the one *obtained in the absence of a magnetic field.* This means that appears in Eq. (8), dt_{BKD}/dz, where $t_{BKD}(z)$ is the temperature profile obtained by Birikh [5–7] (for a mathematical demonstration, see [8]). Thus, the energy equation is still given by (7), and the momentum equation (8) becomes when using Eq. (9)

$$\frac{d^3u_0}{dz^3} = -Ra - Ra_m\left(\cos\phi\sin\phi + \frac{dt_{BDK}}{dz}\sin^2\phi\right). \tag{10}$$

Due to the linearity of the ODE system (7) and (10), the velocity $u_0(z)$ and the temperature $t_0(z)$ are $u_0(z) = u_{BKD}(z) + u_1(z)$, $t_0(z) = t_{BKD}(z) + t_1(z)$, the sum of the solution u_{BDK}, t_{BKD} obtained at $Ra_m = 0$, already derived for a lateral temperature gradient [5–7], plus another term $u_1(z)$, $t_1(z)$ solution of

$$\frac{d^5t_1}{dz^5} = \frac{d^3u_1}{dz^3} = -Ra_m\left(\cos\phi\sin\phi + \frac{dt_{BDK}}{dz}\sin^2\phi\right). \tag{11}$$

When $Ra_m = 0$, Eqs. (7) and (10) lead to

$$u_{BKD}(z) = -\frac{Ra}{12}z(z-1)(2z-1)$$
$$t_{BKD}(z) = \frac{u_{BKD}(z)}{60}(3\,z^2 - 3z - 1)\,. \tag{12}$$

The velocity profile u_{BKD} has two extrema given by $\pm Ra/72\sqrt{3}$, at $z = (1/2)(1 \pm \sqrt{1/3})$. Now, we solve the system obtained from Eqs. (7) and (11) when $Ra_{\text{m}} \neq 0$. The zero order velocity profile $u_0(z)$ is proportional to $u_{\text{BDK}}(z)$ since

$$\frac{u_0(z)}{u_{\text{BDK}}(z)} = 1 + \frac{Ra_{\text{m}}}{Ra} \sin\phi \cos\phi - \frac{Ra_{\text{m}} \sin^2\phi}{2520} (3z^4 - 6z^3 + 3z + 1) . \qquad (13)$$

The polynom $(3z^4 - 6z^3 + 3z + 1)/2520$ is a positive function whose maximum value is $1.9375/2520 \approx 8 \times 10^{-4}$ at $z = 0.5$, the layer middle. We can thus safely use a simplified version of the velocity profile,

$$u_0(z) \approx u_{\text{BDK}}(z) \left(1 + \frac{Ra_{\text{m}}}{Ra} \sin\phi \cos\phi \right) , \qquad (14)$$

provided that $|1 + (Ra_{\text{m}}/Ra) \sin\phi \cos\phi| \gg 8 \times 10^{-4} Ra_{\text{m}} \sin^2\phi$. The velocity $u_0(z)$ is the solution (12) multiplied by the quantity $\mathcal{I} = 1 + (Ra_{\text{m}}/Ra) \times \sin\phi \cos\phi$. The maxima of $u_0(z)$, given by Eq. (14) is

$$\text{Max}(u_0(z)) = (2Ra + Ra_{\text{m}} \sin(2\phi))/144\sqrt{3} \quad \text{at} \quad z_{\text{max}} = 0.788 . \qquad (15)$$

But $\text{Max}(u_0(z))$ is still a function of the inclination and varies between, a maxima maximorum given by $(2Ra + Ra_{\text{m}})/144\sqrt{3}$ at $\phi = \pi/4$, and a minima at $(2Ra - Ra_{\text{m}})/144\sqrt{3}$ at $\phi = 3\pi/4$ (see Fig. 2a). From Eqs. (7), (10) and (12) one obtains finally $t(z) \approx t_0(z)$ where

$$t_0(z) = t_{\text{BDK}}(z) \left(1 + \frac{Ra_{\text{m}}}{Ra} \sin\phi \cos\phi \right) \qquad (16)$$

which is the solution of the energy balance (7) using the approximation $u_0(z)$ given by Eq. (14). The temperature profile is again the product of t_{BKD} by the factor $\mathcal{I}$, like Eq. (14). We can satisfy ourselves with $u_0(z)$, $t_0(z)$, given by Eqs. (14) and (16) to describe the reference motion, since we obtained an excellent agreement between those solutions and exact numerical solutions using the continuation software AUTO97 [14] computed in the space of parameters defined by Table 1 (for an extensive discussion, see [8]).

6. Microgravity level and the factor $\mathcal{I}$

Let us study the multiplying factor $\mathcal{I}$ that combines the gravity level, the saturation magnetization, the temperature gradient and the obliticity of the external field. If $Ra > Ra_{\text{m}}$, whatever the value of $\sin(\phi)\cos(\phi)$, then $0 < \mathcal{I} \leq 1 + Ra_{\text{m}}/Ra < 2$. The inclination can compress or dilate slightly the Birikh profile.

But when $Ra \leq Ra_{\text{m}}$, and when ϕ belongs to the second or the fourth quadrant, $\mathcal{I}$ can become negative. This can lead to a significative difference from

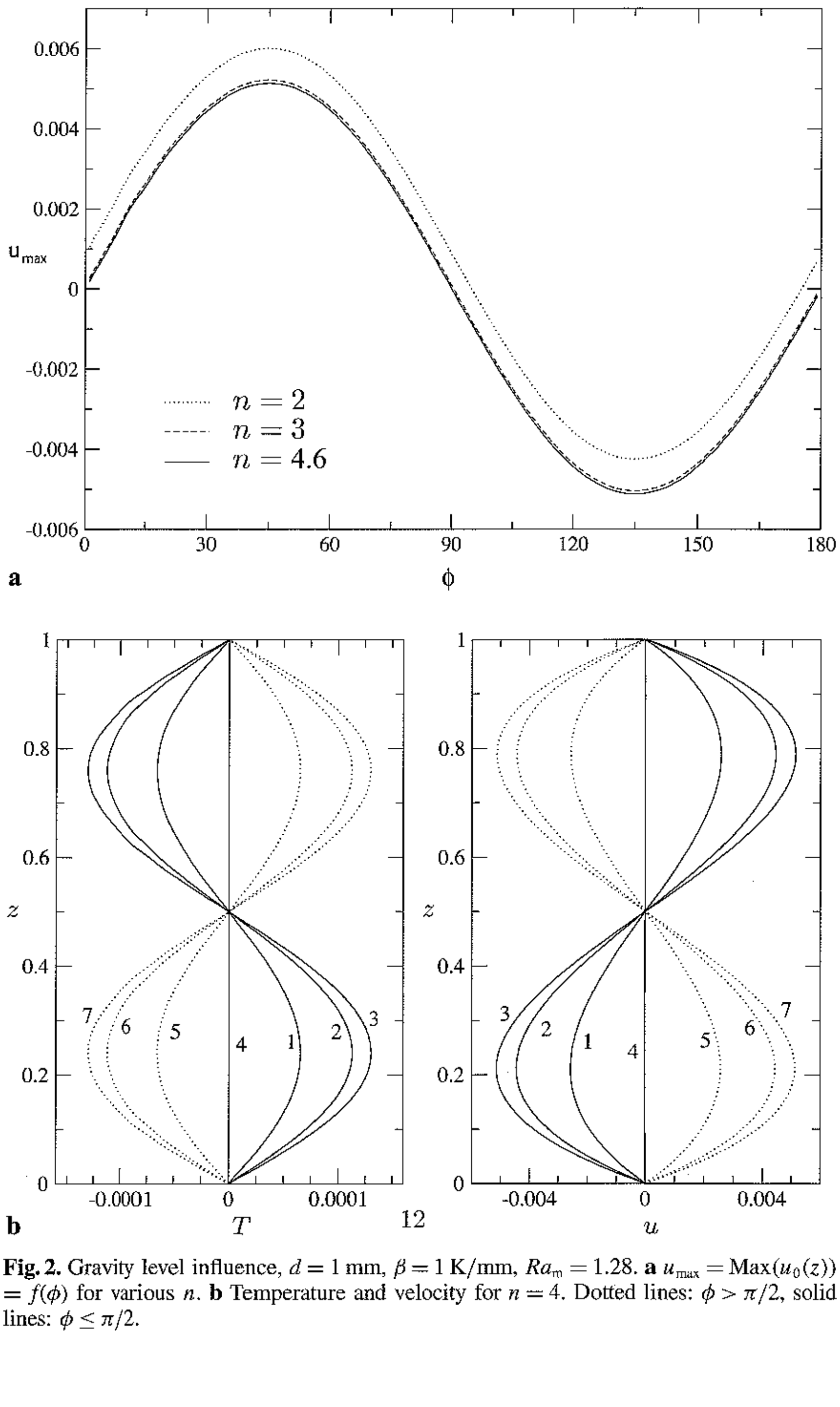

Fig. 2. Gravity level influence, $d = 1$ mm, $\beta = 1$ K/mm, $Ra_m = 1.28$. **a** $u_{max} = \text{Max}(u_0(z)) = f(\phi)$ for various n. **b** Temperature and velocity for $n = 4$. Dotted lines: $\phi > \pi/2$, solid lines: $\phi \leq \pi/2$.

the solutions (12), since even the case $\mathcal{I}=0$ becomes a possibility. Calling y the square of either $\sin\phi$ or $\cos\phi$, one has $\sin(\phi)\cos(\phi)=\pm\sqrt{y(1-y)}$, so that $\mathcal{I}=0$ becomes equivalent to $y^2-y+[Ra/Ra_{\rm m}]^2=0$, whose roots are given by:

$$y_\pm=\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{Ra^2}{Ra_{\rm m}^2}}\quad\text{with}\quad y_++y_-=1\,. \tag{17}$$

The roots $y_\pm$ are physically meaningful only when they are real positive quantities, less than 1. The following condition must be satisfied: $Ra_{\rm m}\geq 2Ra$. For $y>y_+$ or $y<y_-$, $\mathcal{I}$ is always positive. But, $\mathcal{I}<0$ for $y_-<y<y_+$ and $\mathcal{I}=0$ for a certain value of y given by y_+ or y_-.

This is a specific consequence of having an *inclined* external field. When $\mathcal{I}$ is negative, we have again a convective solution. However, *the convective motion is in the opposite direction of the original classical Birikh one.* The lateral temperature gradient induces a pressure push through mean of gravity, opposed to the one due to the Kelvin force that becomes sufficiently strong to overcome the gravitational force and change the direction of the return flow.

The physically admissible value of y_+ defines a certain angle $\phi_{\rm cr}=\pi/2+\theta_{\rm cr}$ in the second quadrant. Since $1=y_++y_-$, exists another motionless heat conductive solution of Eq. (17) at $\pi-\theta_{\rm cr}$. The same will apply in the fourth quadrant. For those two specific inclinations, $u_0(z)=t_0(z)=0$ for all z, and the solution is a motionless conductive one as the temperature reduces to $T=T_{\rm ref}-\beta\,{\rm d}x$ *across the whole layer*. Physically, both lateral pressure gradient compensate each other.

To illustrate this, let us consider EMG901 (see Table 1). The ratio of $Ra_{\rm m}/Ra\geq 10^n/8.59$ must be larger than two. Any microgravity level 9.81×10^{-n} with n such that $n\geq\log_{10}(8.59\times 2)\approx 1.29$, will be sufficient. In Fig. 2a, we follow the maximum velocity (15) of (14) as a function of ϕ for various microgravity levels starting from $n=2$, which corresponds to the gravity level in a parabolic flight, $n=4$ the one in a drop tower or in sounding rockets, down to $n=6$ the one obtained in a Foton satellite. As n increases, microgravity decreases so that buyoancy becomes negligibly small. Thus $\mathcal{I}\approx\sin(2\phi)Ra_{\rm m}/2Ra$, $y_+=1$, $y_-=0$ and ${\rm Max}(u_0(z))\approx Ra_{\rm m}\sin(2\phi)/144\sqrt{3}$ becoming practically independant upon n as Fig. 2a records when $n\geq 4$. For $n\geq 4$, the maximum of the velocity ${\rm Max}(u_0(z))$ is strictly proportional to $\sin(2\phi)$. Let us increase the angle ϕ, then ${\rm Max}(u_0(z))$ increases many times more than the initial value at $\phi=0$ until it reaches a maximum maximorum at $\phi=\pi/4$ where it decreases towards zero at $\phi=\pi/2$, where we do not observe any motion. Increasing the obliquity still, the absolute value of the maxima starts to increase again, but in the opposite direction and the profile of velocity reaches a negative extrema at $\phi=3\pi/4$ and then decreases again towards its original value at $\phi=\pi$. For $n<4$, we observe exactly the same trend, with very slight differences. We can observe that also in Fig. 2b. The profiles of $u_0(z)$ and $t_0(z)$ are depicted for $n = 4$, which is equivalent to reducing gravity by a factor of ten thousand, so that Ra vary now like $0.0011d^4$ while $Ra_m = 1.28d^4$ and the ratio $Ra_m/Ra \approx 1{,}164$ whatever the depth d. The factor $\mathcal{I}$ is positive for $0 \leq \phi < \pi/2$, equal to zero at $\phi = \pi/2$ and negative for $0 \leq \phi < \pi/2$. Curves 1, 2, 3 represent the profiles of temperature and velocity at $\phi = 15°$, $\phi = 30°$ and $\phi = 45°$ respectively. They are symetrical to Curves 5, 6, 7 which correspond to $\phi = 165°$, $\phi = 150°$ and $\phi = 135°$. Curve 4 at $\phi = \pi/2$ shows a heat conductive rest state.

7. Conclusion

We have considered here a thin layer of a ferrofluid EMG 901, heated laterally and submitted to an inclined magnetic strong field. The profiles of velocity or of temperature are the product of two quantities (see Eqs. (14) and (16)). The first one is the laminar profiles of velocity and heat, for a Newtonian fluid submitted to a lateral gradient [5]. The second one is $\mathcal{I} = 1 + (Ra_m/Ra)\sin\phi\cos\phi$, which is independent upon the depth d of the layer and the applied magnetic field amplitude H_0^e.

When Ra_m/Ra is large enough, the role of buoyancy and magnetic field are reversed. Buoyancy is a small term, with respect to the magnetic field which is largely dominant. The inclination of the magnetic field can give rise to a pure conductive solution at two critical angles $\phi = \pi/2 + \theta_{cr}$ and $\phi = \pi - \theta_{cr}$ for which $\mathcal{I} = 0$. As Figs. 2a and b show, $\theta_{cr} = 0$, for large n, thus as buyoancy due to microgravity is decreased. Then, for any inclination belonging to the second quadrant, the direction of the motion and the temperature profile are symmetric with respect to the one in the first quadrant. In a microgravity environment, an inclined magnetic field has therefore drastically modified the reference steady state of a liquid layer heated laterally. The very simple expression derived in this work for the velocity and the temperature profiles should induce future work on the stability of that base state.

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References

1. J. Huang, B. D. Edwards, and D. D. Gray, Phys. Fluids **9** (1997) 1919.
2. D. Braithwaite, E. Baugnon, and R. Tournier, Nature **354** (1991) 134.
3. B. A. Finlayson, J. Fluid Mech. **40** (1970) 753.
4. S. Odenbach, Magnetohydrodynamics **38** (2002) 239.

5. R. V. Birikh, J. Appl. Mech. Tech. Phys. **3** (1966) 69.
6. A. G. Kirdyashkin, Int. J. Heat Mass Transfer **7** (1984) 1205.
7. S. H. Davis, Ann. Rev. Fluid Mech. **19** (1987) 406.
8. M. Hennenberg, B. Weyssow, S. Slavtchev, V. Aleksandrov, T. Desaive, and B. Scheid, submitted to Phys. Fluids.
9. R. E. Rosensweig, *Ferrohydrodynamics*. Dover Publ., N.Y. (1997).
10. V. G. Bashtovoi, B. M. Berkovski, and A. N. Vislovich, *Introduction to Thermomechanics of Magnetic Fluids*. Hemisphere Publ. Corp., N.Y. (1988).
11. T. Völker and S. Odenbach, Phys. Fluids **15** (2003) 2198.
12. J. Weilepp and H. R. Brand, J. Phys. II France **5** (1996) 419.
13. E. Blums, A. Cebers, and M. M. Maiorov, *Magnetic Fluids*. de Gruyter, Berlin (1997).
14. E. Doedel, A. Champneys, T. Fairfrieve, Y. Kuznetsov, B. Sandstede, and X. Wang, AUTO 97: Continuation and bifurcation software for ordinary differential equations. Tech. Rep. Department of Computer Science, Concordia University, Montreal, Canada (1997). (Available from ftp.cs.concordia.ca in directory pub/doedel/auto)